Suggested solution of HW3

Chapter 2 Q11: (a) By scaling, we assume $R = 1$. Consider $g = f \circ \phi$ where $\phi(z) = \dfrac{z+a}{1+\bar{a}z}$.

By Cauchy formula, we get

$$f(a) = g(0) = \frac{1}{2\pi i}\oint_{\partial B(1)} \frac{g(z)}{z}\,dz = \frac{1}{2\pi i}\oint_{\partial B(1)} \frac{f\circ\phi(z)}{z}\,dz.$$

We now perform change of coordinate $w = \phi(z)$. We have

$$\frac{dz}{dw} = \frac{1-|a|^2}{(1-\bar{a}w)^2} \neq 0 \quad \text{and} \quad z = \frac{w-a}{1-\bar{a}w}.$$

Also, ϕ map $\partial\mathbb{D}$ to $\partial\mathbb{D}$ bijectively. Thus,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\oint_{\partial B(1)} \frac{f(w)}{w-a}\cdot\frac{1-|a|^2}{1-\bar{a}w}\,dw \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta}-a}\cdot\frac{1-|a|^2}{1-\bar{a}e^{i\theta}}\cdot e^{i\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})(1-|a|^2)}{(e^{i\theta}-a)(e^{-i\theta}-\bar{a})}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})Re\left(\frac{e^{i\theta}+a}{e^{i\theta}-a}\right)d\theta. \end{aligned}$$

(b) Follows from direct computation.

Chapter 2 Q12: (a) Let $g(z) = 2\dfrac{\partial u}{\partial z}$. Since u is harmonic,

$$\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z}u = 0 \implies \frac{\partial}{\partial\bar{z}}g = 0.$$

Hence, $g(z)$ is holomorphic. As $\mathbb{D}$ is simply connected, g has a primitive F on $\mathbb{D}$, $\dfrac{\partial F}{\partial z} = g(z) = 2\dfrac{\partial u}{\partial z}$. Since F is analytic, using the fact that $\frac{\partial F}{\partial \bar{z}} = 0$, we have

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = 2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

From $\partial_x F = \partial_x Re(F) + i\partial_x Im(F)$, we know that

$$\partial_x\left[Re(F) - u\right] = 0$$

and similarly from $\partial_y F$,

$$\partial_y\left[Re(F) - u\right] = 0.$$

That show that $Re(F) = u + C$ for some real constant C (viewing it as a function on $\mathbb{R}^2$). By considering $F - C$, we may assume $C = 0$. Suppose f_1, f_2 are two analytic functions so that

$$Re(f_i) = u.$$

Then $h(z) = f_1(z) - f_2(z)$ is analytic and by CR equation

$$\nabla(Im(h)) = \Big(\partial_x Im(h), \partial_y Im(h)\Big) = 0.$$

Therefore, they only differ by a real constant.

(b) By part (a), there exists a holomorphic function f such that $Re(f) = u$. Applying the formula in Q11 on a slightly smaller ball $B(r)$ and takes real part on both side. The result follows by taking $r \to 1$.

Remark : One can also show it by proving the Poisson integral formula agrees with u on the boundary. By Maximum Principle, u agree with the Poisson formula in the interior as well. It can be seen that the integral representation of f in Q11 can also be obtained from the Poisson formula of its real part and imaginary part.

Chapter 3 Q15: (a) Writting $f(z) = \sum_{n=0}^{\infty} a_n z^n$. For $n > k$, by Cauchy formula,

$$a_n = \frac{1}{2\pi i} \oint_{\partial B(R)} \frac{f(z)}{z^{n+1}}\, dz.$$

Thus,

$$|a_n| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{AR^k + B}{R^n}\, d\theta \to 0 \text{ as } R \to \infty.$$

Thus, f is a polynomial of degree at most k

(d) By translation and scaling, we can assume $f(\mathbb{C}) \subset \{z : 0 \leq Re(z) \leq 1\}$. Consider $g(z) = e^f$, g is a bounded entire function. Thus g is constant implying f is also a constant function.

Chapter 3 Q19: (a) Assume u attain a maximum at p in Ω, there exists $r > 0$ such that $B(p, r) \subset \Omega$. By previous exercise, there exists a holomorphic function f on $B(p, r)$ such that $Re(f) = u$. But it contradicts with open mapping theorem. So it is impossible.

(b) It is a direct consequence of part (a).

Remark : One can also prove it directly using the mean value property (Strong Maximum Principle) of harmonic function.

Chapter 5 Q4: (a)

$$F(z) = \prod_{n=1}^{N-1} (1 - e^{-2\pi nt} e^{2\pi iz}) \cdot \prod_{n=N}^{\infty} (1 - e^{-2\pi nt} e^{2\pi iz})$$

Consider the series $\sum_{N+1}^{\infty} |e^{-2\pi nt} e^{2\pi iz}|$. Choose $N \approx |z|/t$ yields

$$\sum_{n=N}^{\infty} |e^{-2\pi nt} e^{2\pi iz}| \leq e^{-2\pi Im(z)} \sum_{n=N}^{\infty} e^{-2\pi nt} \leq e^{2\pi|z|} \frac{e^{-2\pi tN}}{1 - e^{-2\pi t}} \leq \frac{1}{1 - e^{-2\pi t}}.$$

Thus, $\prod_{n=N}^{\infty}(1-e^{-2\pi nt}e^{2\pi iz})$ is bounded above by a constant C_t. Now we estimate the term with finite elements.

$$\prod_{n=1}^{N-1}(1-e^{-2\pi nt}e^{2\pi iz}) \leq (1+e^{2\pi|z|})^N \leq 2^N e^{2\pi|z|N} \leq e^{c_1|z|+c_2|z|^2}$$

for some constant $c_1, c_2 > 0$. So F is of order ≤ 2.

Now we claim the reverse inequality. It can be observed that $m - int$ are the zeros of F. And for $p \leq 2$,

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{|m-int|^p} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m^2+n^2t^2)^{p/2}} = +\infty$$

by comparing this with $\displaystyle\int_1^{\infty}\int_1^{\infty}\frac{1}{(x^2+t^2y^2)^{p/2}}dx\,dy$.

An alternative way is to estimate $M(r)$, we argue as follows.

For $|z| = r$, where r is sufficiently large. Writing $iz = x + iy$, denote $e^{-2\pi nt}$ by R, it can be checked that

$$|1-e^{-2\pi nt}e^{2\pi iz}|^2 = 1-2Re^{2\pi x}\cos y + R^2e^{4\pi x}.$$

So, the maximum is at least $Re^{2\pi\sqrt{r^2-\pi^2}}$ when r is large enough. Denote $M(r)$ to be the maximum of $F(z)$ on $B(r)$. We deduce that

$$M(r) > \prod_{n=1}^{\infty}(1+e^{-2\pi nt}e^{2\pi\sqrt{r^2-\pi^2}}) \approx \prod_{n=1}^{\infty}(1+e^{-2\pi nt}e^{2\pi r}).$$

Taking log gives,

$$\log M(r) \geq \sum_{n=1}^{\infty}\log(1+e^{-2\pi nt}e^{2\pi r}) = \sum_{n=1}^{N}\log(1+e^{-2\pi nt}e^{2\pi r}) + \sum_{n=N+1}^{\infty}\log(1+e^{-2\pi nt}e^{2\pi r})$$

Since $\log(1+z) \geq z/2$ whenever $0 < z < 1/2$, the second terms is bounded below by C_t if we choose $N \approx (\log 2 + 2\pi r)/2\pi t$.
On the other hand,

$$\begin{aligned}\sum_{n=1}^{N}\log(1+e^{-2\pi nt}e^{2\pi r}) &\geq \int_1^N \log(1+e^{2\pi r}e^{-2\pi xt})\,dx \\ &> \int_1^{N/2}\log(1+e^{2\pi r}e^{-2\pi xt})\,dx \\ &= x\log(1+e^{2\pi r}e^{-2\pi xt})\Big|_1^{N/2} + \int_1^{N/2}\frac{2\pi x\cdot e^{2\pi r}}{e^{2\pi xt}+e^{2\pi r}}\,dx.\end{aligned}$$

Noted that for $x \in [1, N/2]$, $\dfrac{e^{2\pi r}}{e^{2\pi x t} + e^{2\pi r}}$ is bounded below by $1/4$ when $r \to \infty$. So we have for some constant $C = C(t) > 0$,

$$\sum_{n=1}^{N} \log(1 + e^{-2\pi n t} e^{2\pi r}) \geq \int_1^N \log(1 + e^{2\pi r} e^{-2\pi x t})\, dx$$

$$> \frac{\log 2 + 2\pi r}{4\pi t} \log\left(1 + \frac{e^{\pi r}}{\sqrt{2}}\right) + \frac{\pi}{4}\left[\frac{\log 2 + 2\pi r}{4\pi t}\right]^2 - C.$$

Combining all the inequalities, we can see that $\underline{\lim}_{r\to\infty} \frac{\log M(r)}{r^2} > \tilde{C}$ for some constant $\tilde{C} > 0$. Thus, the order is at least 2.

(b) $F(z) = 0$ iff one of its factors is 0. So, it vanishes exactly when $z = -int + m$ for $n \geq 1$, m, n are integers.

Chapter 5 Q5:

$$F_\alpha(z) = \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi i z t}\, dt$$

Noted that

$$|e^{2\pi i z t}| = e^{-2\pi t Im(z)} \leq e^{2\pi |t||z|} \leq e^{|t|^\alpha/\alpha} \cdot e^{C_1 |z|^{\alpha/\alpha - 1}}$$

where C_1 is a constant obtained from Young ‘s inequality.

$$F_\alpha(z) = \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi i z t}\, dt \leq e^{C_1|z|^{\alpha/\alpha-1}} \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{|t|^\alpha/\alpha}\, dt = C_2 e^{C_1|z|^{\alpha/\alpha-1}}.$$

So, $Ord(f) \leq \alpha/\alpha - 1$.
To show the reverse inequality, we make use of the Jensen's inequality.

$$i.e. \quad \log\left[\int_a^b f(x)\, dx\right] \geq \frac{1}{b-a}\int_a^b \log((b-a)f(x))\, dx, \quad \forall \text{ integrable } f \geq 0.$$

Put $iz = x \in \mathbb{R}$, for any $R > 0$

$$\log F_\alpha(z) \geq \log\left[\int_0^\infty e^{-t^\alpha} e^{2\pi x t}\, dt\right] \geq \log\left[\int_0^R e^{-t^\alpha} e^{2\pi x t}\, dt\right]$$

$$\geq \frac{1}{R}\int_0^R \log[R \cdot e^{-t^\alpha} e^{2\pi x t}]\, dt = \log R + \frac{1}{R}\int_0^R (2\pi x t - t^\alpha)\, dt$$

$$= \log R + \pi x R - \frac{R^\alpha}{\alpha + 1}.$$

Choose R such that $\pi x = \frac{2}{\alpha+1} R^{\alpha-1}$ implying

$$\log F_\alpha(z) \geq \log R + \frac{1}{\alpha+1} R^\alpha = \frac{1}{\alpha - 1}\log x + C^\alpha x^{\frac{\alpha}{\alpha-1}} + \log C$$

where $C = (\frac{\alpha+1}{2}\pi)^{1/(\alpha-1)}$.
This show that the order of F_α is at least $\alpha/(\alpha - 1)$.

Chapter 5 Q10: (a)

$$\begin{aligned} e^z - 1 &= e^{z/2}\left(e^{z/2} - e^{-z/2}\right) \\ &= 2e^{z/2}\sinh(z/2) \\ &= -i2e^{z/2}\sin(iz/2) \end{aligned}$$

But we have

$$\sin z = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 n^2}\right).$$

So

$$e^z - 1 = -i2e^{z/2}(\frac{iz}{2})\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{4\pi^2 n^2}\right) = e^{z/2}z\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

(b) Since we have

$$\sin 2z = 2z\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{\pi^2 n^2}\right)$$

and

$$\sin 2z = 2\sin z\cos z$$

We deduce that

$$\begin{aligned} \cos z &= \frac{1}{2}\cdot 2z\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{\pi^2 n^2}\right)\cdot\left[z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 n^2}\right)\right]^{-1} \\ &= \prod_{n\text{ is odd.}}\left(1 - \frac{4z^2}{\pi^2 n^2}\right) = \prod_{n=0}^{\infty}\left(1 - \frac{4z^2}{\pi^2(2n+1)^2}\right). \end{aligned}$$

Chapter 5 Q11: By multiplication and substraction, we can assume f miss 0 and 1. By Hadamard's theorem, we have

$$f(z) = e^{P(z)} = e^{Q(z)} + 1$$

for some poly $P(z)$ and $Q(z)$ with same degree. If degree is non-zero, by fundamental theorem of algebra, there exists z_0 such that $Q(z_0) = i\pi$. Thus, $e^{P(z_0)} = 0$ which is impossible. So both $P(z)$ and $Q(z)$ are constant function, implying f is constant.

Chapter 5 Q14: Assume F has finitely many zeros. By Hadamard's factorization theorem,

$$F(z) = e^{P(z)}z^m\prod_{n=1}^{N}E_k(z/a_n)$$

for some $a_1, a_2, ...a_N$, where P is a polynomial of degree $k < \rho$. But then F is of order k instead of ρ. Contradict with our assumption.

Chapter 5 Q15: Suppose f has poles at $\{a_n\}_{n\in\mathbb{N}}$ counting with multiplicity. Let g be a entire function having zeros precisely at $\{a_n\}$ by weierstrass factorization theorem. Hence $f(z)g(z)$ is entire.

Similarly, let $f(z), g(z)$ be entire functions having zeros precisely at $\{a_n\}$ and $\{b_n\}$ respectively. Then the function $f(z)/g(z) = h(z)$ is the desired entire function.

Chapter 6 Q5: Put $s = 1/2 + it$ into the product formula of $\Gamma(s)$, we have

$$\Gamma(1/2+it)\Gamma(1/2-it) = \frac{\pi}{\sin \pi(1/2+it)}.$$

But $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt$, so $\overline{\Gamma}(z) = \Gamma(\bar{z})$.

$$\begin{aligned}\Gamma(1/2+it)\overline{\Gamma(1/2+it)} &= \Gamma(1/2+it)\Gamma(1/2-it)\\ &= \frac{\pi}{\sin \pi(1/2+it)}\\ &= \frac{2\pi}{e^{\pi t}+e^{-\pi t}}.\end{aligned}$$

Chapter 6,Q12: (a) Noted that for all $s \in \mathbb{C}$,

$$\frac{1}{\Gamma(s)} = \frac{\sin \pi s}{\pi}\Gamma(1-s).$$

Put $s = -k - 1/2$, gives

$$\frac{1}{|\Gamma(s)|} = \frac{1}{\pi}\left|\Gamma(k+3/2)\right|.$$

By the functional equation of Γ, we get

$$\Gamma(k+1)\Gamma(k+3/2) = \sqrt{\pi}2^{-2k-1}\Gamma(2k+2) = \sqrt{\pi}2^{-2k-1}(2k+1)!.$$

Thus,

$$\begin{aligned}\frac{1}{|\Gamma(s)|} &= \frac{1}{\pi}\left|\Gamma(k+3/2)\right|\\ &= \frac{(2k+1)!}{\sqrt{\pi}\cdot k!2^{2k+1}}\\ &= \frac{1}{2\sqrt{\pi}}\frac{2k+1}{4}\cdot\frac{2k}{4}\cdots\frac{k+1}{4}\\ &\geq \frac{1}{2\sqrt{\pi}}\left(\frac{k}{4}\right)^k.\end{aligned}$$

Taking log, one can observe that

$$\log\frac{1}{|\Gamma(s)|} \geq C_1 + k\log k - k\log 4.$$

So the order of growth cannot be $O(e^{c|s|})$.

(b) If we can find such entire function F. Clearly, the order of growth of F is 1. By Hadamard's factorization theorem,

$$F(z) = e^{Az+B} z \prod_{n=1}^{\infty} E_1(-\frac{z}{n}) = e^{Az+B} z \prod_{n=1}^{\infty} (1+\frac{z}{n}) e^{-z/n}.$$

Thus,

$$\frac{1}{\Gamma(z)} = e^{-B} F(z) e^{z(-A+\gamma)}.$$

The right hand side is of order $O(e^{c|z|})$ while left hand side not. So contradiction occurred.

Extra question: If $a \in (0,1]$, we consider the contour γ from $c - iR \to c + iR$ and followed by a semi-circular arc L of radius R centered at c in clockwise direction. Therefore,

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{a^z}{z(z+1)} \, dz = 0.$$

On the other hand, as $0 < a \leq 1$, $|a^z| \leq a^{Re(z)} \leq 1$ on L.

$$\left| \int_L \frac{a^z}{z(z+1)} \, dz \right| \leq O(R) \cdot O(\frac{1}{R^2}) \to 0 \quad \text{as} \quad R \to \infty.$$

Result follows.

If $a > 1$, consider the contour γ from $c - iR \to c + iR$ and followed by a semi-circular arc L of radius R centered at c in anti-clockwise direction.

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{a^z}{z(z+1)} \, dz = Res(f, 0) + Res(f, -1) = 1 - \frac{1}{a}.$$

Argue as first case, we know that

$$\left| \int_{L \cap \{Re z \leq 0\}} f(z) \, dz \right| \to 0 \quad \text{as} \quad R \to \infty.$$

On $L \cap \{Re z > 0\}$, by symmetry, the total integral is 0. Or you can use a better contour consisting of $\{x + iR : x \in [0, c]\}$, $\{x - iR : x \in [0, c]\}$ and a semi-circlar arc centered at 0 with radius R. The computation is almost same.